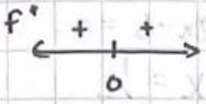
AP Calculus BC

Concavity / P.o.I. / Second Derivative

1) $f(x) = 2x^4 - 8x$



$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$

$x = 0$

f has no P.o.I → f" does not Δ signs.

2) $f(x) = (2x^2+5)^3$

$f'(x) = 3(2x^2+5)^2 \cdot 4x = 12x(2x^2+5)^2$

$f''(x) = 12x[2(2x^2+5) \cdot 4x] + 12(2x^2+5)^2 = 0$

$= 12(2x^2+5)[2x^2 + 2x^2+5] = 0$

$= 12(2x^2+5)(4x^2+5) = 0$

$f''(x) \neq 0$

f has no P.o.I.

3) $f(x) = e^{-x^2}$

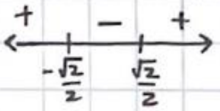
$f'(x) = -2xe^{-x^2}$

$f''(x) = 4x^2e^{-x^2} - 2e^{-x^2} = 0$

$= e^{-x^2}(4x^2-2) = 0$

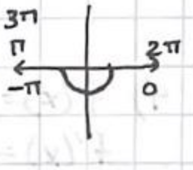
$4x^2 - 2 = 0$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$



f(x) has P.o.I @ $x = \pm\frac{\sqrt{2}}{2}$
b/c f" Δs signs

4) $f(x) = \sin(\frac{1}{2}x)$

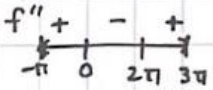


$f'(x) = \frac{1}{2}\cos(\frac{1}{2}x)$

$f''(x) = -\frac{1}{4}\sin(\frac{1}{2}x) = 0$

$\sin(\frac{1}{2}x) = 0$

$\frac{1}{2}x = 0, \pi, 2\pi,$



$x = 0, 2\pi, \cancel{4\pi}$

f has P.o.I. @ $x = 0, 2\pi$
f" Δs signs

5) $f''(x) = x^2(x-4)(x-7) = 0$

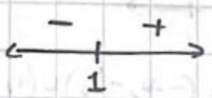
$x = 0 \quad x = 4 \quad x = 7$

f has P.o.I @ x = 4, x = 7
b/c f"(x) Δs signs

6) $g(x) = \frac{x}{x-1}$ * infinite disc @ x = 1

$g'(x) = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2} = -(x-1)^{-2}$

$g''(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$



g is concave down on $(-\infty, 1) \to f'' < 0$
g is concave up on $(1, \infty) \to f'' > 0$

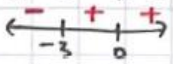
7) $f(x) = x^5 + 5x^4$

$f'(x) = 5x^4 + 20x^3$

$f''(x) = 20x^3 + 60x^2 = 0$

$20x^2(x+3) = 0$

$x = 0 \quad x = -3$



* f has P.o.I @ x = −3
f" Δs signs.

8) $f(x) = 2x^3 - 3x^2 - 12x + 18$

$f'(x) = 6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2 \quad x = -1$

f' $\quad + \quad - \quad +$

$\quad -1 \quad 2$

f'' $\quad - \quad +$

$\quad \frac{1}{2}$

$f''(x) = 12x - 6 = 0$

$x = \frac{1}{2}$

f'' $\quad - \quad +$

$\quad \frac{1}{2}$

f is increasing And concave down on $(-\infty, -1)$ b/c $f' > 0$ & $f'' < 0$

9) $f(x) = 4x^3 + ax^2 + bx + k$

$f'(x) = 12x^2 + 2ax + b$

$f'(-1) = 0$

$12(-1)^2 - 2a + b = 0$

$-2a + b = -12$

$f''(x) = 24x + 2a$

$f''(-2) = 0$

$-48 + 2a = 0$

$\boxed{a = 24}$

$-48 + b = -12$

$\boxed{b = 36}$

10) $f'(2) < f(2) < f''(2)$

11) a) $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

$f'(x) = 12x^3 - 12x^2 - 12x + 12$

$f''(x) = 36x^2 - 24x - 12 = 0$

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$x = -\frac{1}{3} \quad x = 1$

$+ \quad - \quad +$

$-\frac{1}{3} \quad 1$

- f is concave up on $(-\infty, -\frac{1}{3}) \cup (1, \infty)$ $f'' > 0$
- f is concave down on $(-\frac{1}{3}, 1)$ - $f'' < 0$
- f has P.o.I @ $x = -\frac{1}{3}, 1 \rightarrow f''$ Δs signs

b) $f(x) = \arcsin x \quad [-1, 1]$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)$

$= \frac{x}{(1-x^2)^{3/2}}$

$x = 0$

$- \quad +$

$-1 \quad 0 \quad 1$

f is concave up on $(0, 1) \rightarrow f'' > 0$

f is concave down on $(-1, 0) \rightarrow f'' < 0$

f has P.o.I. @ $x = 0 \rightarrow f''$ Δs signs.

c) $f(x) = \cos^2 x - 2\sin x$ $[0, 2\pi]$

$f'(x) = -2\cos x \sin x - 2\cos x$

$= -2\cos x(\sin x + 1)$

$f''(x) = -2\cos x(\cos x) + (\sin x + 1)(2\sin x)$

$= -2\cos^2 x + 2\sin^2 x + 2\sin x$ $\cos^2 x + \sin^2 x = 1$

$= -2(1 - \sin^2 x) + 2\sin^2 x + 2\sin x$ $\cos^2 x = 1 - \sin^2 x$

$= -2 + 2\sin^2 x + 2\sin^2 x + 2\sin x = 0$

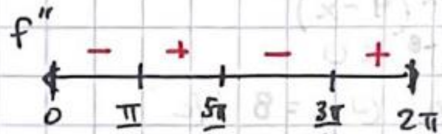
$4\sin^2 x + 2\sin x - 2 = 0$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $x = \frac{3\pi}{2}$



- $f(x)$ is concave up on $(\frac{\pi}{6}, \frac{5\pi}{6})$ $(\frac{3\pi}{2}, 2\pi)$ b/c $f'' > 0$
- $f(x)$ is concave down on $[0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{3\pi}{2})$ b/c $f'' < 0$
- $f(x)$ has D.o.I @ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ b/c f'' Δs signs

12) a) $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0$ $x = 2$

$f''(x) = 6x - 6$

$f''(0) = -6$ $f''(2) = 6$

- $f(x)$ has max @ $x = 0$ b/c $f'(0) = 0$ ∮ $f''(0) < 0$
- $f(x)$ has min @ $x = 2$ b/c $f'(2) = 0$ ∮ $f''(2) > 0$.

b) $f(x) = 2x^2 \ln x$

$f'(x) = 2x^2(\frac{1}{x}) + 4x \ln x$

$= 2x + 4x \ln x = 0$

$2x(1 + 2\ln x) = 0$

~~$x = 0$~~ $\ln x = -\frac{1}{2}$

Not in Domain $x = \frac{1}{\sqrt{e}} = e^{-1/2}$

$f''(x) = 2 + 4x(\frac{1}{x}) + 4\ln x$

$= 6 + 4\ln x$

$f''(e^{-1/2}) = 6 - 2 = 4$

$f(x)$ has min @ $x = e^{-1/2}$ b/c $f'(e^{-1/2}) = 0$

$f''(e^{-1/2}) > 0$

c) $f(x) = e^{-x}(x-7)$

$f'(x) = e^{-x} + (x-7)(-e^{-x}) = 0$

$e^{-x}(1-(x-7)) = 0$

$e^{-x}(-x+8) = 0$

$x = 8$

$f''(x) = e^{-x}(-1) + (-x+8)(-e^{-x})$

$= -e^{-x} - e^{-x}(8-x)$

$= -e^{-x}(9-x)$

$f''(8) = -e^{-8} < 0$

* f has max @ $x = 8$ b/c $f'(8) = 0$ & $f''(8) < 0$

d) $f(x) = x + 2\sin x$ on $(0, 2\pi)$

$f'(x) = 1 + 2\cos x = 0$

$\cos x = -1/2$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$f''(x) = -2\sin x$

$f''(\frac{2\pi}{3}) = -2(\frac{\sqrt{3}}{2}) = -\sqrt{3}$

$f''(\frac{4\pi}{3}) = -2(-\frac{\sqrt{3}}{2}) = \sqrt{3}$

* f has max @ $x = \frac{2\pi}{3}$ b/c $f'(\frac{2\pi}{3}) = 0$ & $f''(\frac{2\pi}{3}) < 0$

* f has min @ $x = \frac{4\pi}{3}$ b/c $f'(\frac{4\pi}{3}) = 0$ & $f''(\frac{4\pi}{3}) > 0$.

13) $\frac{dy}{dx} = 4x + y$

$\frac{d^2y}{dx^2} = 4 + \frac{dy}{dx}$

$= 4 + 4x + y$

In Quad 1, $\frac{d^2y}{dx^2} > 0$

$\therefore$ y is concave up

14) $y = x^3 - 3x^2 - 4$

$y' = 3x^2 - 6x$

$y'' = 6x - 6 = 0$

$x = 1$

$y(1) = -6 \quad y'(1) = -3$

$(1, -6)$

$\boxed{y + 6 = -3(x-1)}$

15) $\frac{dy}{dx} = 6 - 2y$

a) $\left.\frac{dy}{dx}\right|_{(0,4)} = -2$

$L(x) = 4 - 2(x - 0)$

$f(0.6) \approx L(0.6) = 4 - 2(0.6)$

$= 4 - 0.3 = 3.7$

b) $\frac{d^2y}{dx^2} = -2\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{(0,4)} = 4 > 0$

$L(0.6) < f(0.6)$